State-variable biquads with optimum integrator sensitivities

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Abstract: We show how to derive state-variable biquadratic sections with lowest possible sensitivity to their integrators. The resulting structures turn out *to* satisfy the condition for optimum dynamic range given by Mullis and Roberts [1]. The sensitivity optimum obtained is very 'strong' in the sense that these biquads simultaneously attain lower bounds for several practical measures of sensitivity. Furthermore, it is shown that this class of filters exhibits sensitivities that are either equal to or lower than those of doubly-terminated *LC* ladders.

1 Introduction

This paper discusses a particular application of a novel transfer-function synthesis technique described by Snelgrove and Sedra [2]. Although this synthesis method may be applied to both continuous-time and discrete-time networks, we concentrate in this paper on the continuous-time case. However, the results can easily be related to some other work in discrete-time filters [1,3].

Many of the basic ideas of this paper are applicable to transfer functions of arbitrary degree, but we concentrate our attention here on the 2nd-order case. The general case is an interesting and useful area for research, and the results given here for biquads suggest analogies for higher orders.

We present the various topologies in terms of signal-flow graphs, which are readily converted to either active-RC or switched-C circuits, so that the underlying structures are clearly visible.

The biquad topologies derived have transfer functions that are equally sensitive to each of the two integrators in the circuit, a situation which minimises several important measures of overall sensitivity. As will be shown, these equalised sensitivity functions are proportional to the derivative of the transfer function $[dT(s)/ds]$, a condition which we relate to Orchard's observation on the sensitivity of doubly-terminated passive filters with maximum power transfer [4].

2 Synthesis from intermediate transfer functions

We show in Reference 2 how one can synthesise a unique state-space structure that realises a given overall transfer function T(s) by choosing, rather than a topology, a set of ***intermediate transfer function*** $f_i(s), i = 1,2, \ldots, N$ where N is the order of T(s). These functions become the transfer functions from the filter input to the outputs of the N integrators in the state-space realisation. Since the only requirements for this set of functions are that they be linearly independent and that they have the same poles as those of T(s), considerable freedom exists in choosing an appropriate set. This freedom can be used to optimise the sensitivity and dynamic range of the resulting realisation. In this paper, we use sensitivity identities to find the set of intermediate transfer functions which result in optimum sensitivity realisations of biquadratic filters.

In addition to the f functions mentioned above, we will also be interested in a dual set of transfer functions – those from the integrator inputs to the filter outputs. We shall denote these latter functions g_i. The set f_i uniquely determines the dual set g_i, and vice-versa.

3 Sensitivity relationships

One may show [2] that the sensitivity of a transfer function T(s) to the gain γ_i of the ith integrator of a system is

$$S_{\gamma i}^{T(s)} \triangleq \frac{\gamma_i}{T(s)} \frac{dT(s)}{d\gamma_i} = f_i g_i \frac{s}{T(s)} \tag{1}$$

One may also show that, for any realisation whatever, the sum over all integrators of these sensitivities is a constant*

$$\sum_i S_{\gamma i}^{T(s)} = -\frac{s}{T(s)} \frac{dT(s)}{ds} \tag{2}$$

This identity derives from the fact that the change of all integrator gains by a given percentage simply frequency shifts the overall filter response by the same percentage, regardless of topology; the reason some topologies are better than others is that they all behave differently when integrator gains change by different amounts. We can choose to evaluate the performance of a design with such measures as

$$\int W(\omega) \sum_i |S_{\gamma i}^{T(j\omega)}|^2 d\omega \qquad \text{(a statistical measure)}$$

and

$$\max_{\omega \in (a,b)} \sum_i |S_{\gamma i}^{T}(j\omega)| \qquad \text{(a worst-case measure)}$$

where $W(\omega)$ is a weighting function.

Now the interesting thing about eqn. 2 is that it implies that one may minimise both of the figures of merit above, and many more, by forcing all integrator sensitivities to be equal. In general, a constraint of the form

$$\sum_i x_i = k$$

where k is a constant, results in the choice $x_1 = x_2 = \ldots = k/N$, minimising any function of the form

$$\sum_i |x_i|^p \qquad p \geqslant 1$$

as well as functions such as

$$\sum_i |\mathrm{Re}\,(x_i)|^p$$

Thus, choosing all sensitivities equal,

$$S_{\gamma i}^{T(s)} = -\frac{s}{NT(s)} \frac{dT(s)}{ds} \qquad \forall_i \tag{3}$$

*This corresponds to a familiar identity in passive network theory, concerning the sum of sensitivities to all reactive components.

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yields a system that simultaneously optimises most reasonable sensitivity measures!

Note that the equality we desire is between sensitivity *functions* of ω: sensitivities should ideally be equal (both in sign and magnitude) at all frequencies. This design objective is sometimes attainable and sometimes not: for biquads in particular, it usually turns out to be possible.

4 Optimum sensitivity biquads

Combining eqns. 1 and 3 results in the following condition on the f and g functions of state-space realisations that have optimum sensitivity:

$$f_i g_i = -\frac{1}{N}\frac{dT(s)}{ds} \qquad \forall_i \tag{4}$$

Unfortunately, the synthesis method does not allow us to select $(f_i g_i)$ but only f_i (or g_i). Thus, for the optimum realisation to exist, we have to find a set of linearly independent functions f_i which must all divide $[dT(s)/ds]$, as eqn. 1 implies, and whose dual set g_i results in the condition of eqn. 4 being satisfied.

To understand how all of this works, we shall consider the case of a 2nd-order lowpass filter

$$T(s) = \frac{a}{s^2 + s\left(\dfrac{\omega_0}{Q}\right) + \omega_0^2} \triangleq \frac{a}{E(s)}$$

The derivative can be easily found to be

$$\frac{dT(s)}{ds} = -\frac{a\left(2s + \dfrac{\omega_0}{Q}\right)}{E(s)^2}$$

Now, we know from eqn. 4 that $f_1(s)$ and $f_2(s)$ must divide $[dT(s)/ds]$, and that they must be linearly independent. Choosing

$$f_1(s) = 1/E(s)$$

and

$$f^*(s) = \left(2s + \frac{\omega_0}{Q}\right)/E(s)$$

satisfies both conditions. In fact (apart from trivial changes such as scale factors and interchange of f_1 and f_2), this is the only choice possible. Using the results from Reference 2, we can now obtain the signal-flow graph realisation of Fig. 1 and evaluate the g functions as

$$g_1 = \frac{a}{2}\left(2s + \frac{\omega_0}{Q}\right)/E(s)$$

$$g_2 = \frac{a}{2}/E(s)$$

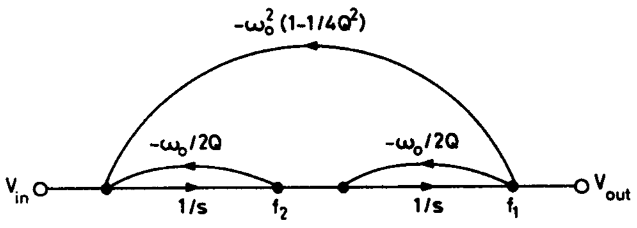

Fig. 1 *Optimum LP biquad*

We thus see that

$$f_1 g_1 = f_2 g_2 = -\frac{1}{N}\frac{dT(s)}{ds}$$

which is the condition for optimum sensitivity. Thus, the realisation of Fig. 1 is the optimum sensitivity realisation of the 2nd order lowpass function. Note that, for this structure,

$$f_1 = \tfrac{1}{2} a g_2$$

and

$$f_2 = \tfrac{1}{2} a g_1$$

This can be shown to give the condition derived in Reference 3 for a 2nd-order system to have optimum dynamic range (after L_2 scaling, which does not affect sensitivities). In fact, biquads with this 'reciprocity' property always have equal integrator sensitivities. If

$$g_1 = \alpha f_1$$

and

$$g_2 = \alpha f_1$$

then

$$f_1 g_1 = f_2 g_2$$

We thus conclude that optimum integrator sensitivity produces optimum dynamic range in biquads.

The same procedure can be applied to bandpass, highpass, allpass and notch biquad transfer functions, and so produces **optimum** integrator sensitivity structures for most of the interesting biquads. It should be mentioned, however, that for certain transfer functions (primarily those with very low-Q poles) it is not possible to fmd two suitable factors of $[dT(s)/ds]$ for f_1 and f_2.

5 Relationship to LC ladders

In general, an Nth-order filter with equal integrator sensitivities is insensitive to integrator gain errors at the reflection zeros (i.e. frequencies of maximum transmission) ω_r because

$$S_{\gamma i}^{|T(j\omega_r)|} = \mathrm{Re}\,[S_{\gamma i}^{T(j\omega_r)}]$$

$$= \mathrm{Re}\left[\frac{-j\omega_r}{N}\frac{1}{T(j\omega)}\frac{dT(j\omega)}{d(j\omega)}\right]_{\omega=\omega_r}$$

$$= \mathrm{Re}\left[-\frac{\omega_r}{N}\frac{d\ln T(j\omega)}{d\omega}\right]_{\omega=\omega_r}$$

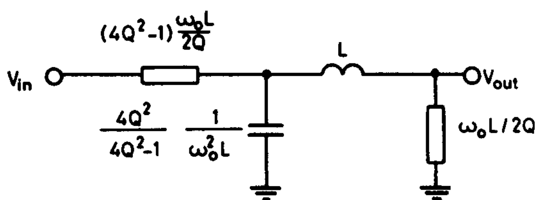

Fig. 2 LP ladder

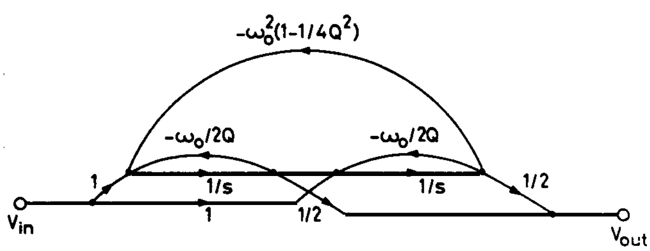

Fig. 3 *Optimum BP biquad*

Table 1: Sensitivity comparison: ladder against optimum

Circuit	$f_1 g_1$	$f_2 g_2$
Fig. 3	$(s^2 - \omega_0^2)/2E(s)^2$	$(s^2 - \omega_0^2)/2E(s)^2$
Fia. 4	$-\omega_0^2/E(s)^2$	$s^2/E(s)^2$

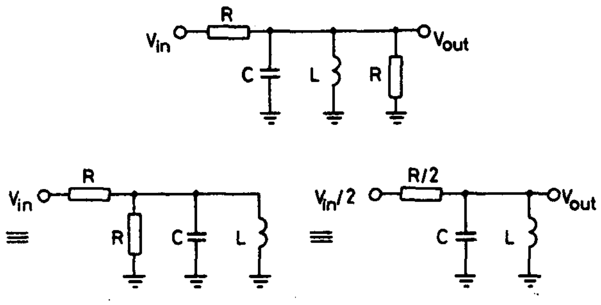

Fig. 4 *Equivalent ladders*

Expressing $T(j\omega)$ *as* .

$$T(j\omega) = |T(j\omega)|e^{j\phi(\omega)}$$

results in

$$S_{\gamma i}^{|T(j\omega_r)|} = \frac{-\omega_r}{N} \mathrm{Re}\left[\frac{d\ln|T(j\omega)|}{d\omega} + j\frac{d\phi(\omega)}{d\omega}\right]_{\omega=\omega_r}$$

But since $d\ln|T(j\omega)|/d\omega|_{\omega=\omega_r} = 0$, it follows that

$$S_{\gamma i}^{|T(j\omega_r)|} = 0$$

This is the same property that *LC* ladders which are designed for maximum power transfer have. In the following, we show that our optimum biquads either simulate the corresponding *LC* ladders (in which case they have the same sensitivity performance) or are new structures that exhibit lower sensitivities than ladders.

In the signal flow graph of Fig. 1, it turns out that the signals at the two nodes labelled f_1 and f_2 simulate the capacitor voltage and inductor current of the doubly-terminated lowpass ladder in Fig. 2, which has maximum power transfer at its reflection zeros. On the other hand, the signal flow graph of the optimum bandpass biquad shown in Fig. 3 does not simulate a doubly-terminated bandpass ladder. To further investigate this point, we demonstrate in Fig. 4, through a Thévenin equivalence, that for the bandpass case a doubly-terminated realisation is no better than a singly-terminated one. The optimum structure of Fig. 3 may be compared as to sensitivity with one simulating one of the ladders of Fig. 4: the integrator sensitivity products $(f_i g_i)$ are given in Table 1.

It is interesting to note that the two structures have equal sensitivity at the reflection zero $s \pm j\omega_0$, which is where the doubly-terminated structure is known to be good. Our new structure becomes superior away from $j\omega_0$ (according to any aggregate sensitivity measure of the types discussed earlier).

6 Concluding remark

The main objective of this paper has been to outline a technique for obtaining state-space structures with optimum sensitivity. The most striking feature of the structures generated is their physical symmetry, which results from the requirement that they be equally sensitive to all their integrators. We are currently investigating the use of the optimum biquad structures to realise higher-order transfer functions by replacing the integrators with blocks of higher-order transfer functions. These latter blocks should in turn have optimum or ‘good’ sensitivity with respect to their integrators.

7 Acknowledgment

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