Frequency-Domain Analysis of A/D Converter Nonlinearity

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TOPIC: CIRCUITS AND SYSTEM THEORY OR ANALOG CIRCUITS AND FILTERS

ABSTRACT

A novel two-dimensional z-transform approach to the analysis of analog/digital (A/D) converter nonlinearity is presented. It was primarily developed to study the stability behavior of an adaptive self-trimming technique for flash and multi-stage A/D converters, but has more general applicability.

I. INTRODUCTION

An adaptive self-trimming algorithm for A/D converters [1,2] has been presented which continually trims thresholds in the flash A/D subconverters of two-stage and pipelined A/D converters. Roughly speaking, it trims thresholds up or down by small increments in such a way as to smooth out irregularities in the code density of the A/D subconverters' digital output. To demonstrate its stability and predict convergence rate, we developed a novel two-dimensional (2-D) z-transform approach. This paper introduces the 2-D z-transform approach, and illustrates its general applicability in frequency-domain interpretation of the distinct properties of differential and integral nonlinearity in A/D converters and its applicability in describing a simple adaptive threshold-trimming system. The application of this approach for a comprehensive analysis of the practical adaptive self-trimming systems presented in [1,2] is presented in [3].

Figure 1 shows the input-output characteristic of an ideal 2-bit flash analog/digital converter. It is like a staircase, with integer output values and "risers" at threshold values x_i, which are ideally spaced uniformly one "least-significant bit (LSB)" Δ apart. Mathematically, these ideal thresholds are $\hat{x}_i = i\Delta$, and the converter's output is $i = \lfloor v_{in}/\Delta \rfloor$. Several definitions are possible, some allowing negative outputs and most with the staircase shifted $\Delta/2$ vertically or horizontally from that shown in Figure 1, but we have chosen this one because it simplifies the analysis and the extension to the others is routine.

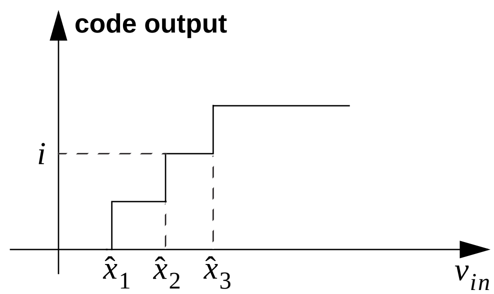


Fig.1 Input/output characteristic of an ideal 2-bit A/D converter

A practical flash A/D converter has its thresholds randomly shifted from the ideal thresholds $\hat{x}_i$, resulting in A/D converter nonlinearity. Two measures commonly used are **differential nonlinearit**y (DNL), which tends to emphasize fine-scale errors, and **integral nonlinearit**y (INL), which shows up cumulative errors affecting the overall shape of the characteristic function.

Mathematically, a practical flash A/D converter produces a digital output

$$D(\mu) = i, \mu \in [x_i, x_{i+1}) \tag{1}$$

DNL is defined as

$$DNL_i = x_{i+1} - x_i - \Delta \tag{2}$$

and INL is deviation of a threshold x_i from its ideal case $\hat{x}_i$, which can also be written as a running sum (hence the word "integral") of DNL

$$INL_i = x_i - i\Delta = \sum_{j=0}^{i} DNL_j \tag{3}$$

An N-bit flash A/D converter only define $2^N - 1$ thresholds, x_1 up to $x_{2^N - 1}$. End-points $x_0 \equiv 0$ and $x_{2^N} \equiv 2^N \Delta$ are defined so that an ideal converter has zero INL and DNL everywhere. We will be paying a lot of attention to this function in the rest of this paper, so we'll give it the name $\gamma_i = INL_i$ for notational convenience (accordingly, $\hat{\gamma}_i$ as its ideal case). As an example, Figure 2 shows the input-output characteristic of a 2-bit flash AID converter with a 0.5Δ threshold error in x_2. Its INL has $\gamma_2 = 0.5$ and is otherwise zero.

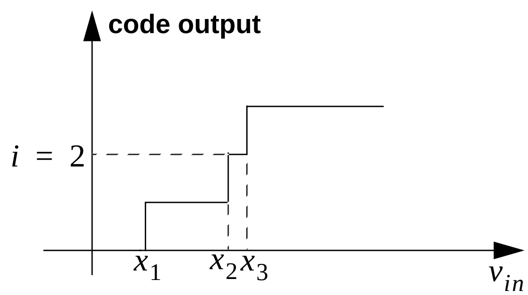


Fig.2 Characteristic of an A/D converter with a threshold error

II. Z-TRANSFORM IN THE CODE DOMAIN

The set of A/D converter thresholds x_i can be regarded as a signal sequence, and then described by a generic z-transform [4]. Generally,

$$Z\{x_i\} = \sum_{i=1} x_i z^{-i} \tag{4}$$

For example, the ideal 2-bit flash AID converter with a set of thresholds $\{x_1, x_2, x_3\} = \{1, 2, 3\}$ has $X(z) = z^{-1} + 2z^{-2} + 3z^{-3}$. The coefficients of the equation have a "ramp" form, so there is an equivalent form with a denominator having a double pole at $z = 1$:

$$X(z) = \frac{1}{(1 - z^{-1})^2}(z^{-1} - 4z^{-4} + 3z^{-5}) \tag{5}$$

The numerator in Eq.(5) contains large terms in z^{-4} and z^{-5} that come from having finite N. The z-transform of INL is more convenient: ideally it's $\Gamma(z) = Z\{\hat{\gamma}_i \equiv 0\} = 0$, and in practice it's a polynomial of order $2^N - 1$ in z^{-1} with small coefficients corresponding to individual threshold errors. DNL is just the "derivative" of INL

$$Z\{DNL_i\} = (z - 1)\Gamma(z) \tag{6}$$

For the example shown in Figure 2, $Z\{DNL_i\} = 0.5z^{-1} - 0.5z^{-2}$.

The important point of using such a z-transform is that it gives us a definition of "frequency", with which we can distinguish between "high-frequency" errors (which model local variation and dominate quantization noise performance) and "low-frequency" errors (which model variations of the overall characteristic from linearity and which dominate low-order harmonic distortion). Evaluating the z-transform of INL at an angle θ on the unit circle gives us

$$\Gamma(e^{j\theta}) = \sum_{i=1}^{2^N - 1} \gamma_i e^{ij\theta} \tag{7}$$

which correlates errors with sinusoids in code i. For example, at $\theta = (2\pi)/(2^N - 1)$ the real and imaginary parts of $e^{ij\theta}$ run through a single cycle each of *co*s and *si*n, and a large $Im\Gamma(e^{j\theta})$ shows that the overall characteristic has a shape like that shown in Figure 3. This would produce significant third-order inter-modulation distortion when converting a large signal.

Notice that, by differentiating Eq.(7), we can derive the z-transformed DNL, and DNL naturally emphasizes its high-frequency components.

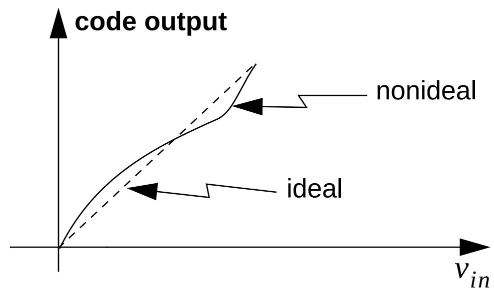


Fig.3 Characteristic of an A/D converter with a low-frequency threshold error

III. 2-D z-TRANSFORM TO DESCRIBE TRIMMING ALGORITHM

We are ultimately interested in error-correction algorithms that trim flash AID thresholds during operation, so that the thresholds x_i become functions of time, i.e., $x_i(t)$. Expressing time variable t in terms of sample number k, we want to understand the behavior of $x_i(k)$. We can then use z-transform techniques on these threshold variables to see whether a trim algorithm can drive INL and DNL to zero, and how quickly.

Thresholds x_i are now functions of two variables, output code i and time k. Thus we need a two-dimensional z-transform to describe them. As before, it is convenient to focus on INL $\gamma_i(k)$:

$$\Gamma(z_c, z_t) = Z_{c,t}\{\gamma_i(k)\} = \sum_{k=0}^{\infty} \sum_{i=1}^{2^N - 1} \gamma_i(k) z_c^{-i} z_t^{-k} \qquad (8)$$

where the two variables z_c and z_t transform the time and code domains respectively. For example, if there is a simple offset error that drops geometrically to zero with time, thresholds will be $x_i = i\Delta + \varepsilon^k$ and errors $\gamma_i(k) = \varepsilon^k$. The z-transform of the errors $\gamma_i(k)$ is

$$\Gamma(z_c, z_t) = (z_c^{-1} + z_c^{-2} + \ldots + z_c^{-(2^N - 1)})(1 + \varepsilon z_t^{-1} + \varepsilon^2 z_t^{-2} + \ldots) \tag{9}$$

and the term in z_t can be rewritten as $1/(1 - \varepsilon z_t^{-1})$ (as long as $|\varepsilon| < 1$).

Now the final-value theorem [4] tells us that a trimming algorithm can eventually drive errors to

$$\lim_{z_t \to 1} (z_t - 1)\Gamma(z_c, z_t) \tag{10}$$

and, for the trivial case of Eq.(9), the final error is zero. In general, unfortunately, two-dimensional transforms will not have the convenient separable form of Eq.(9), but we will still be able to make use of them (see beflow and [3]).

Let us now look at a more useful example in which a trim algorithm adaptively drives flash A/D thresholds towards mean of neighbors. The trim algorithm can be mathematically expressed as:

$$x_i(k) = x_i(k-1) + \mu[x_{i+1}(k-1) - 2x_i(k-1) + x_{i-1}(k-1)] \quad \text{, at time } kT \tag{11}$$

with two fixed end-points at: $x_0(k) \equiv 0$ and $x_{2^N}(k) \equiv V_{ref}$ for all k, where μ is a small-value adaptation step size. For simplicity, we use a 2-bit flash A/D converter as our illustrative example, for which the threshold sequence $\hat{x}_i(k)$ in the ideal case can be written as:

$$\hat{x}_i(k) = 0, 1, 2, 3, 4LSB$$

A set of state-space equations for this adaptive 2-D system can thus be written in matrix form as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1-2\mu & \mu & 0 \\ 0 & 1-2\mu & \mu \\ 0 & \mu & 1-2\mu \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4\mu \end{bmatrix} \tag{12}$$

or $\dot{X} = AX + B$. We can examine the stability of this system by evaluating the eigenvalues of the characteristic function $det[\lambda I - A] = 0$. The eigenvalues of this function can be readily found as:

$$\lambda_1 = 1 - 2\eta$$

$$\lambda_{2,3} = 1 - 2\eta \pm \sqrt{2}\eta, \text{ where } \eta > 0 \quad (13)$$

The eigenvalues are actually the poles of the state-space system in the z_t plane. From Eq. (13), we can see that regardless of the value of η, these poles are always all located inside the unit circle, i.e., the stability of this system is guaranteed. When $\eta = 0.1$, for example, we have $\lambda_1 = 0.8$, $\lambda_2 = 0.9414$, and $\lambda_3 = 0.6586$. The conclusion from such a 2-bit flash A/D converter applies to a larger system, since the larger system has a larger matrix A but the regular structure of the matrix remains the same.

This adaptive 2-D system can be readily analyzed by the 2-D z-transform. As discussed above, we focus on INL $\gamma_i(k)$ which gives our favoured boundary condition: $\gamma_i(k) \equiv 0$ for $i \notin [1, 2^N - 1]$. In terms of $\gamma_i(k)$, Eq.(11) can be rewritten as:

$$\gamma_i(k) = \gamma_i(k-1) + \mu[\gamma_{i+1}(k-1) - 2\gamma_i(k-1) + \gamma_{i-1}(k-1)] \quad (14)$$

The 2-D z-transform of Eq.(14) is

$$\Gamma(z_c, z_t) = \frac{z_t}{z_t - 1} Z\{\gamma_i(0)\} + \frac{\mu}{z_t - 1}(z_c - 2 + z_c^{-1})\Gamma(z_c, z_t) \quad ,$$

$$\text{where } \Gamma(z_c, z_t) = Z\{\gamma_i(k)\} \quad (15A)$$

and then

$$H(z_c, z_t) = \frac{\Gamma(z_c, z_t)}{Z\{\gamma_i(0)\}} = \frac{z_t}{z_t - 1 - \mu(z_c - 2 + z_c^{-1})} \quad (15B)$$

The transfer function of the 2-D system is not generally separable in terms of z_c and z_t, but for our

A/D converter case this doesn't complicate our analysis much because both the adaptation step size μ and the term $(z_c - 2 + z_c^{-1})$ in the denominator of Eq.(15B) (the latter term reflects the deviation of z_c values from their ideal cases) are always relatively small compared to unity. In this case we can see from Eq.(15B) that the 2-D system behaves as a low-pass (LP) filter in the z_t domain. In the equation, the output $\Gamma(z_c, z_t)$ is the i-th threshold at time k, and the input $Z\{\gamma_i(0)\}$ the initial value of the i-th threshold which exists prior to the adaptation process. The second term of Eq.(15A) represents the threshold nonuniformity of the threshold array, because $(z_c - 2 + z_c^{-1}) = (z_c - 1)^2 z_c^{-1}$ estimates the second derivative. When all three thresholds of the 2-bit A/D converter are perfectly aligned, this term should perish. Otherwise, the second term is expected to compensate the error portion of the initial threshold $\gamma_i(0)$.

The "first"-order LP filter has multiple poles (from the variable z_c) in the z_t plane at:

$$z_t = 1 + \mu(z_c - 2 + z_c^{-1}). \tag{16}$$

The z_c values at equilibrium from the 2-bit A/D converter can be estimated by

$$Z\{\hat{\gamma}_i\} = \sum_{i=1}^{3} \hat{A} z_c^{-i}, \text{ where } \hat{A} \to 0$$

$$= \hat{A} \cdot z_c^{-3}(z_c^2 + z_c + 1) \tag{17}$$

which has the following two ideal zeros:

$$\hat{z}_{c(1,2)} = e^{j\frac{2\pi}{3}}, e^{j\frac{4\pi}{3}}. \tag{18}$$

These zeros are all on the unit circle, as shown in Figure 4 (other detail of the figure is to be explained below).

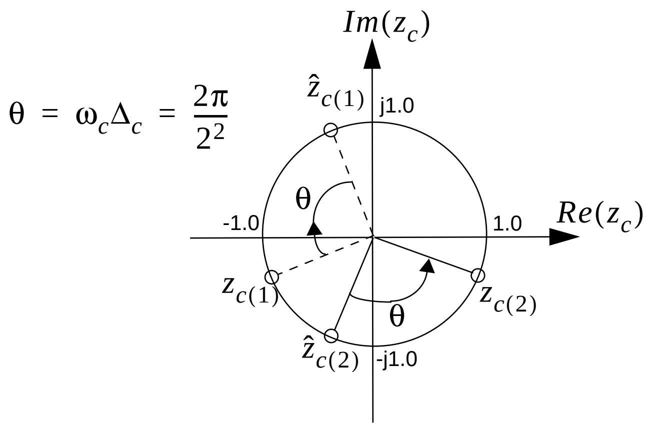


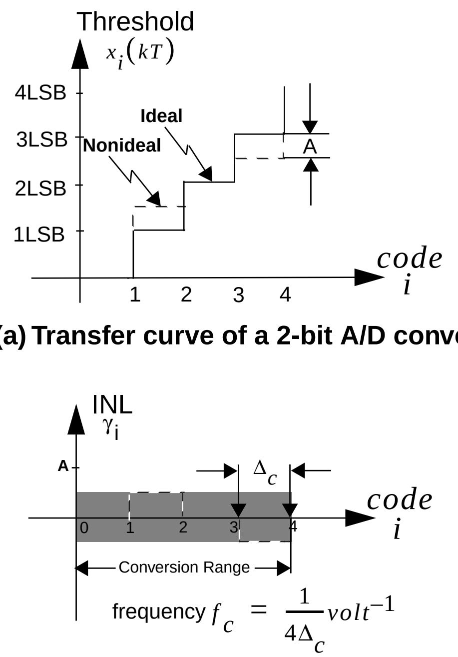
Fig.4 Zero location of an INL sequence

Let us then co … characterized by INL:

$\gamma_i =$

the "sampling" interval $\Delta_c = 1LSB$ (19)

We can define the angular frequency ω_c in Eq.(19) as $\omega_c = 2\pi\frac{M}{4\Delta_c}$ (unit of $\frac{\omega_c}{2\pi}$ = 1/volt) where M is the number of the periods of the INL sequence γ_i. Figure 5 shows the variation of the threshold sequence x_i from its ideal case $\hat{x}_i$ (recall that $\gamma_i = x_i - \hat{x}_i$) when $\alpha = 1$ and $M = 1$, i.e., a low-frequency distortion occurs. In this case we have the z-transform of the INL's periodic complex exponential $\gamma_i = Ae^{j\omega_c\Delta_c i}$:



(a) Transfer curve of a 2-bit A/D converter

(b) Corresponding INL

Fig.5 Integral nonlinearity of a threshold sequence

$$Z\{\gamma_i\} = \sum_{i=1}^{3} A e^{j\omega_c \Delta_c i} z_c^{-i}$$

$$= A \cdot R \cdot z_c^{-3}(z_c^2 + R z_c + R^2), \text{ where } R = e^{j\omega_c \Delta_c} \qquad (20)$$

Two offset zeros for this threshold sequence can be found from Eq.(20):

$$z_{c0(1,2)} = Re^{j\frac{2\pi}{3}}, Re^{j\frac{4\pi}{3}} \qquad (21)$$

which are shown together with the two ideal zeros in Figure 4. The filtering effect of our adaptive trimming technique is to move these offset zeros $z_{c0(1)}$ and $z_{c0(2)}$ to their ideal positions by an angle of $\theta = \omega_c \Delta_c = \frac{2\pi}{2^2}$.

In [3], we apply the 2-D z-transform to analyze a practical histogram-based threshold-trimming system which consists of the core of an on-line self-calibration technique for multi-stage A/D converters presented in [1,2].

IV. CONCLUSION

A novel two-dimensional z-transform approach to the analysis of A/D converter nonlinearity has been presented. The point of using such an approach is that it gives us a definition of "frequency", with which we can distinguish between errors resulting from the local variations of an A/D transfer curve and those from the variation of overall characteristic from linearity. It has been found that integral nonlinearity (INL) is more convenient to use in the frequency-domain analysis. By trimming A/D converter thresholds in some way such as in a simple state-space system illustrated in this paper and those practical systems in [1,2], we can eventually drive integral nonlinearity and differential nonlinearity to zero. The 2-D z-transform can be applied to reveal the functionality and stability behavior of these systems.

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