Stability in a General ΣΔ Modulator

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ABSTRACT

This paper studies the stability of a single-quantizer ΣΔ modulator in terms of its noise transfer function. We show that stability is conditional upon the input signal, and give an explicit formula for the set of all inputs which result in stable behaviour. In addition, we compare the set of modulators stable with zero input to the sets given by several existing rules of thumb and find that these criteria are neither necessary, nor in some cases sufficient, to ensure stability.

Introduction

Sigma-delta modulators are widely used to construct highly linear analog-to-digital converters[1]. They have achieved this prominence mainly because of their tolerance to variations in their analog components. These converters operate by shaping the quantization noise spectrum in such a way that the noise in the band of interest is reduced to the desired level. The digital output requires filtering to remove the out-of-band quantization noise and downsampling to reduce the data rate to the Nyquist rate.

Despite the success of this approach, the basic operation of a sigma-delta modulator is not well understood at the theoretical level. The most pressing theoretical issue is that of stability: under what conditions is a sigma-delta modulator stable? This paper details a general model of a ΣΔ modulator and then investigates the conditions under which it remains stable.

A General Model of a Sigma-Delta Modulator

In a sigma-delta modulator with a single quantizer, the output y may be written as

$$y = g \otimes u + h \otimes e,$$

where u is the input signal, e is the error signal, and g and h are the impulse responses of the signal and noise transfer functions. This equation appears linear, but it is able to exactly describe the nonlinear behaviour of a ΣΔ modulator because it hides the fact that e is a function of u.

The relationship between e and u is best expressed with a diagram. Figure I shows one possible realization of a ΣΔ modulator with an error transfer function H and a signal transfer function G. As shown in the diagram, the error signal is defined as the difference between the output and the input of the quantizer Q, and so it clearly depends on the input.

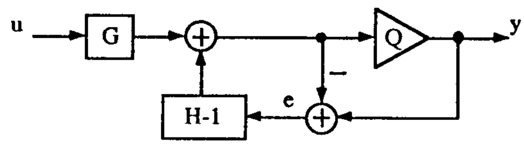

Figure 1: A block diagram of a general ΣΔ modulator with our canonical structure.

Figure 1 is a simple way to construct a ΣΔ modulator with a given H, and we have found analyses based on this diagram to be especially lucid. We therefore advocate this structure for analysis. However, it is believed that this structure is sensitive to errors in the H-l block, and although it is operationally equivalent to any other structure having the same signal and noise transfer functions, the structure of Figure 1 is not recommended for real circuits.

The choice of error transfer function, H, essentially specifies the modulator. Typically the designer selects H to have sufficient attenuation in the band of interest to achieve the desired conversion accuracy. A necessary condition for the realizability of a ΣΔ modulator with a specific H is that H-l be strictly causal, or equivalently, that $h(0)=1$. This constraint is easy to cope with, but one must resort to lengthy simulations to verify that the modulator actually functions as desired. It is the purpose of this paper to shed some light on this important problem.

Simplifications

We begin by setting G=l. This simplification is reasonable because G is not in the loop and consequently it is not expected to affect the stability of the modulator. We shall also focus our attention on modulators with single-bit quantizers, but the ideas may be generalized to multi-bit ones.

Stability is Conditional upon the Input

It is well-known that a ΣΔ modulator may be stable for some inputs but not for others. In this section we present a formula for the set of all inputs which keep a modulator stable.

Firstly, with G=l and u binary* it is easy to see from Figure 1 that $y=u$ and $e=0$ is consistent with the operation of the modulator provided the H-l block is initially at rest. For at time zero, $y(0)=\text{sgn}(u(0))=u(0)$, since u is binary, and so $e(0)=0$. This means that H-l stays at rest and by induction $y=u$ and e=O for all time. The simplicity of this derivation is one reason why we prefer Figure 1 to a more traditional diagram of a ΣΔ modulator.

The fact that a sigma-delta modulator has an output equal to the input under the conditions given above says that sigma-delta modulation is ***idempotent.*** We point out this property here to show that there exists a large class of bounded input signals which keep the loop stable, for any H.

A time-domain analysis of Figure 1 shows that an input u will give rise to an output y if and only if there exists e (e turns out to be the error signal) such that

$$u(n) = y(n) - (h \otimes e)(n) \quad (1)$$

$$\text{and } y(n) = \text{sgn}(y(n) - e(n)) \quad (2).$$

As long as the error signal remains bounded, the modulator is essentially working as desired since e has a finite power and its spectrum is shaped by the noise transfer function. The set of all inputs which ensure $e(n) \in (\text{-M,M}]$ is given by

$$U_M = \{u | u \text{ satisfies the above, and } e(n) \in (-M, M]\}$$

The case $M=1$ is a natural choice since one often assumes in the linear analysis of a ΣΔ modulator that e is uniformly distributed over $(-1,+1]$. In this case, condition (2) is automatically satisfied and we have

$$U_1 = \left\{ \begin{array}{l} u | u(n) = y(n) - (h \otimes e)(n), \\ \text{where } y(n) = \pm 1 \text{ and } e(n) \in (-1,1] \end{array} \right\}$$

These formulae show how stability is linked to the input. Every modulator has inputs for which it is stable in the sense that e remains bounded, and the set of all such inputs has a very special structure. The set consists of non-overlapping regions centered on every possible output pattern. These regions may touch along their edges, or they may be isolated islands. In the case $M=1$, the regions all have the same shape since the $h \otimes e$ term is independent of y. With the insight provided by these formulae, the question of stability becomes a question of whether the sets described above cover a useful range of signals or not.

* By ***binury*** it is meant that the signal is restricted to the two possible outputs of the quantizer: ±1.

Zero-Input Stability

One very important input is zero. This signal is "in the middle" of all the islands of stability and in that sense is equally remote from all of them. As well, any useful set of input signals would likely be centered on zero. It is certainly the case that any useful modulator should be stable with zero input. In this section, we find modulators which are stable with an input of zero.

A ΣΔ modulator is characterized by its noise transfer function. The set of all noise transfer functions is infinite-dimensional and we can only look at a finite-dimensional subset. For the purposes of this paper, it suffices to examine the three-dimensional set: $h=(1,h_1,h_2,h_3,0...)$. For each h chosen from this set, we run a simulation of the modulator corresponding to that h and determine the maximum value of e. Figures 2, 3 and 4 show those modulators wherein $\|e\|_\infty \le 1$ and those wherein $\|e\|_\infty < \infty$ when h_3 is -0.5, 0 and +0.5, respectively.

These pictures show that we are dealing with a very complicated object. There are a few simple criteria mentioned in the literature which are supposed to be able to predict the stability of a ΣΔ modulator, and we shall shortly see that the simple regions they delineate bear only a rough resemblance to the complex object seen here.

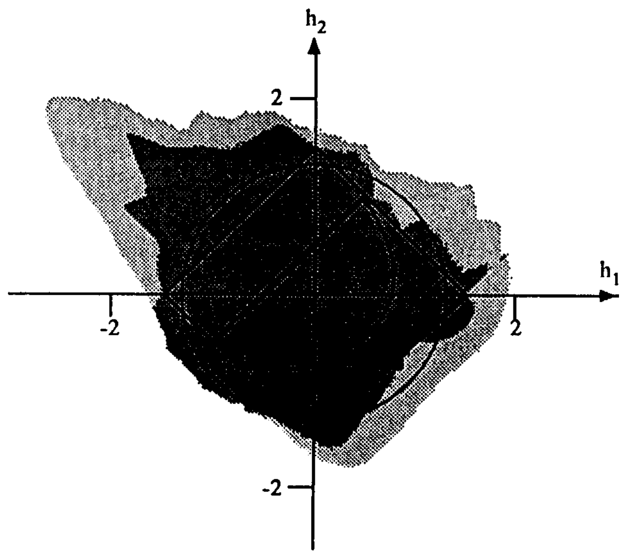

Figure 2: Modulators with h_3=-0.5 and $h_{4...}$=0 which have bounded error signals. In the dark inner region, the error is bounded by unity. The closed figures represent boundaries for several stability criteria mentioned in the literature.

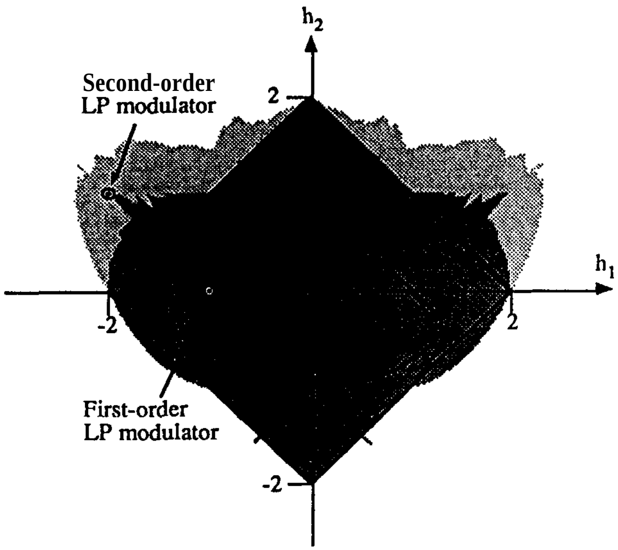

Figure 3: As in Figure 2, except h_3 is now 0. The standard first-order and second-order lowpass modulators are also indicated.

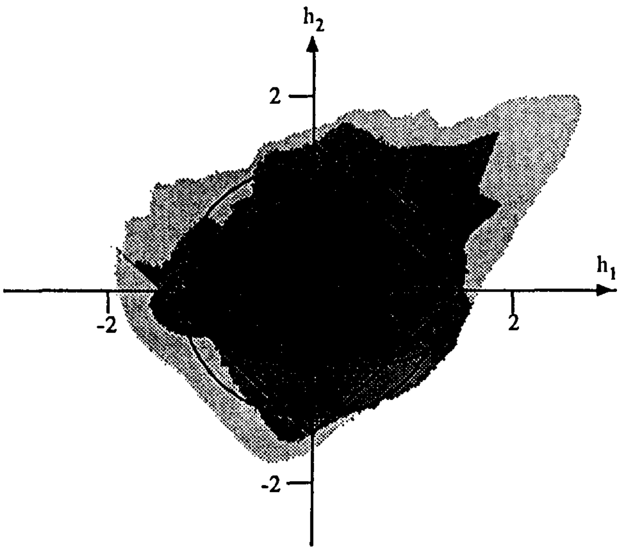

Figure **4:** As in Figure 2, except h_3 is now +0.5. This figure is a reflection about the vertical axis of Figure 2.

Rules of Thumb

The brute force stability test of the previous section requires a modulator to be simulated for infinite time, and even then only guarantees stability if the input is zero. It is clearly desirable to have a shortcut for gauging the stability of a $\Sigma\Delta$ modulator. Several authors have proposed such criteria and we take this opportunity to compare their predictions to the results of the previous section.

The Power Gain Criterion

Agrawal and Shenoi give an argument akin to the following [4]. Since we want the error signal to be white and uniformly distributed over [-1,+1], it must have a power of 1/3. The power in the output signal is 1, so it must be the case that the power gain of the the error transfer function is less than 3. For our parameterization of the error transfer function, this rule of thumb corresponds to a sphere and its circular boundaries are shown on Figures 2-4.

The Maximum Gain Criterion

Lee argues that if the gain of the error transfer function at every frequency is less than 2, the resultant modulator will be stable [3]. The boundary of this region is an irregular curve, possibly with linear segments, and is also plotted in Figures 2-4.

In these figures, the maximum gain criterion appears to be more conservative than the power gain criterion, but in practice it is the other way around. This is true because noise transfer functions designed to satisfy the maximum gain criterion generally have gains near 2 at most frequencies and consequently have a power gain slightly less than 4. This is not apparent in our figures because we are dealing with FIR error transfer functions of extremely low order.

The $\Sigma|h(i)|$ Criterion

Anastassiou mentions that if $\|u\|_\infty \le 1$, then the modulator is guaranteed to be stable with $M=1$ if [2]

$$\sum_{i=1}^{\infty} |h(i)| \le 1.$$

If we restrict the input further, the following simple argument shows that the modulator is stable with $M=1$ if

$$\sum_{i=1}^{\infty} |h(i)| \le 2 - \|u\|_\infty .$$

From Figure 1, we see that if $\|e\|_\infty \le 1$, then the magnitude of the input to the quantizer at time n is

$$\left| \sum_{i=1}^{\infty} h(i)e(n-i) + u(n) \right|$$

$$\le \sum_{i=1}^{\infty} |h(i)e(n-i)| + |u(n)|$$

$$\le \sum_{i=1}^{\infty} |h(i)| + \|u\|_\infty$$

To ensure that $|e(n)| < 1$, it is sufficient to ensure this last term is less than or equal to 2, and a trivial rearrangement of this condition yields the original claim.

Once again, the curves corresponding to this test are plotted on Figures 2-4. Their diamond shape makes them readily identifiable.

Tire Z@(i)~ is the only criterion known by the authors to have a sound theoretical basis. Unfortunately, it can be the most conservative of all. As before, this is not apparent in the pictures because we are dealing with such low-order FIR error transfer functions.

Figures 2-4 clearly show that none of the above criteria completely covers the set of zero-input stable modulators. In particular, Figure 3 shows that none of the stability criteria says that a second-order lowpass modulator, h=(l,-2,-l,O...), ought to be stable, yet it ***is*** stable and is in fact a very popular modulator.

A Counter-Example

Even worse than not being necessary, the power gain and maximum gain criteria are not sufficient to ensure stability. Figure 5 shows a close-up of a part of modulator space wherein completely unstable moduJators pass both criteria.

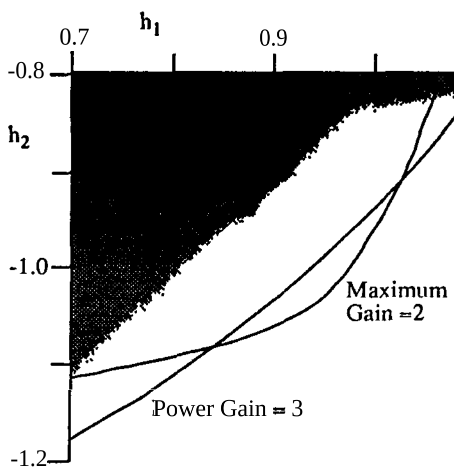

Figure **5:** The stable region of modulator-space with $h_1 \in [0.7,1.1]$, $h_2 \in [-1.2,-0.8]$, $h_3=0.15$, $h_4=-0.3$ and $h_{5...}=0$. Both the power gain and the maximum gain criteria include modulators which are unstable.

Summary

We have studied the problem of stability in a **ΣΔ** modulator characterized by its error transfer function. We found that every modulator is stable for some inputs, but not for others, and we gave a formula for the set of all inputs which keep a particular modulator stable. Although the formula is explicit, we have not been able to use it to determine if a particular modulator is stable for a useful set of input signals.

In search of a practical test for stability, we plotted the zero-input stable modulators in a three-dimensional subspace of modulator space. The resulting object is very complex and consequently has dashed our hopes for finding a simple analytical test for stability in a general moduJator.

Several rules of thumb have been mentioned in the literature, and we compared their simple predictions to the complex object we found. In general, the shapes given by these criteria bear little resemblance to the shape we observe. Much of the region, including a very popular modulator, is not covered by these rules, and in the case of the power gain and maximum gain rules we have found modulators which ought to be stable but are in fact completely unstable.

We extended the rule given by Anastassiou by restricting the input range more severely. This has yielded the most general analytical test for stability, of which we are aware, with a sound theoretical basis. Unfortunately, it remains overly conservative.

References

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